

Chapter 2

Working Stress and Failure Theories

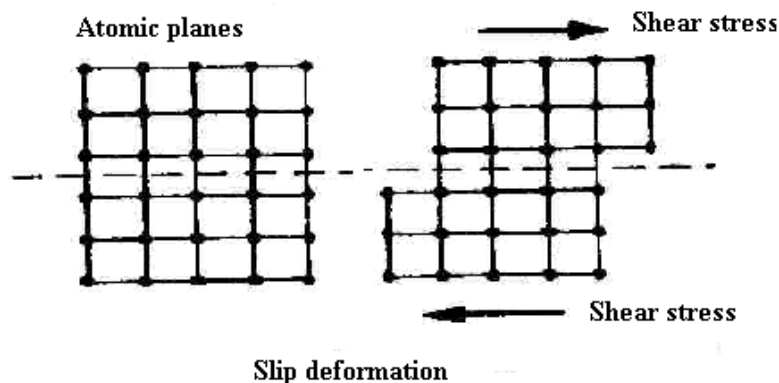
A Simplified Approach

We are interested in learning how **static mechanical stress** can cause failure in machine parts. Static stress means that the stress has been applied slowly and is maintained at a steady level. Failure from **cyclic (or dynamic) stress and impact stress** will be treated later. Here, we should also keep in mind that, there are many other factors such as, surface wear damage from friction, overheating, chemical corrosion, metallurgical fault or a combination of these and other factors may also cause failure.

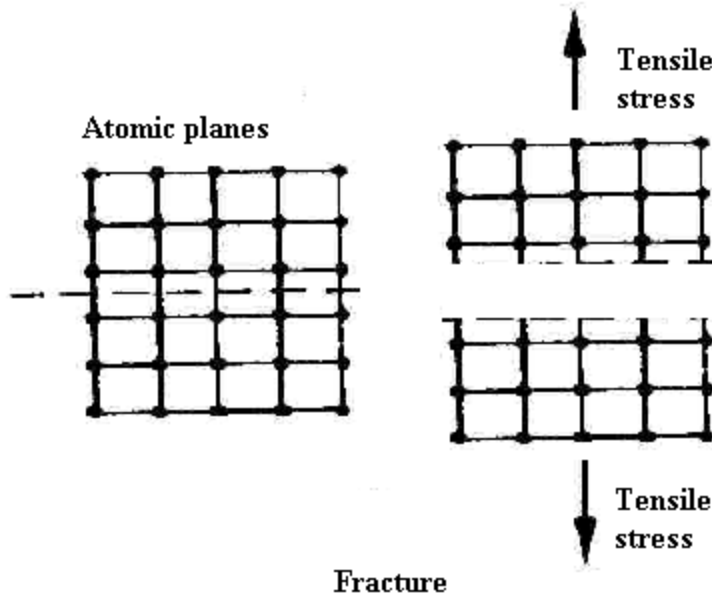
1. Molecular Concept of Mechanical Failure

Engineering materials have a **crystalline molecular structure**, which means the atoms (or molecules) of the material are arranged in a fairly ordered fashion and the atoms are held in fixed position with respect to each other by strong inter-atomic bond. An external mechanical force tends to displace these atoms from their original positions in the direction of the force, which is resisted by the inter-atomic forces. Up to a certain limiting level of the external force, the atoms are displaced to some extent, but are pulled back to their original position when the external force is removed. This phenomenon gives rise to the **elastic behavior** of material (**elastic deformation**), that is up to a certain stress level generally the displacement is proportional to force. Hook's law essentially is the same that is stress is proportional to strain.

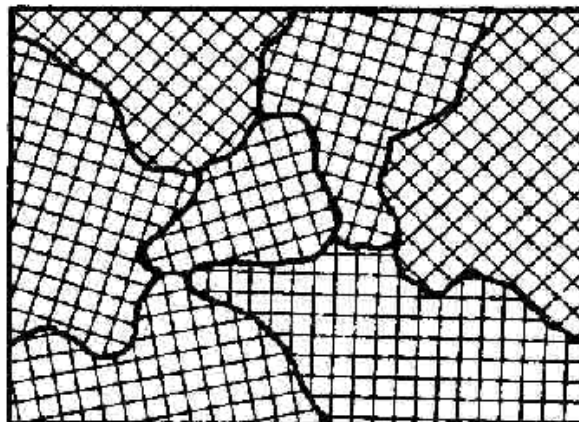
If a force is applied parallel to an atomic plane (**shear force**) and the force is high enough, a plane of atoms may slide over the adjacent plane of atoms overcoming the inter-atomic forces of the immediately neighboring atoms. When this sliding occurs, atoms in the sliding plane will **slip** under the influence of new set of atoms. Conceptual model of this **slip deformation** is shown in the diagram below. After the slip has occurred, the positions of the atoms have changed permanently, resulting in a permanent change in shape or size of the part. This type of permanent deformation is called **plastic deformation**. Plastic deformation is not acceptable in most mechanical design situations, because the permanently deformed part may no longer serve its intended purpose, and from the mechanical design stand point we may say that the part has failed. For example, a landing gear of an aircraft deforms elastically during landing from the ground reaction forces, but we certainly don't want it to be permanently (or plastically) deformed, because then the actuators or other mechanisms may not work properly during the next landing.



If a force applied normally across the atomic plane (**tensile force**) and the force is high enough, two adjacent planes of atoms may separate out from each other producing a crack. The initial crack will reduce the cross-sectional area of the load bearing surface, thus helping the crack to propagate until the entire surfaces separates from the inter-atomic plane. This type of failure, which produces separation of atomic planes, is called **fracture or rupture** type failure. Obviously this type of failure will also not be acceptable in mechanical design. A conceptual model of fracture failure is shown below.



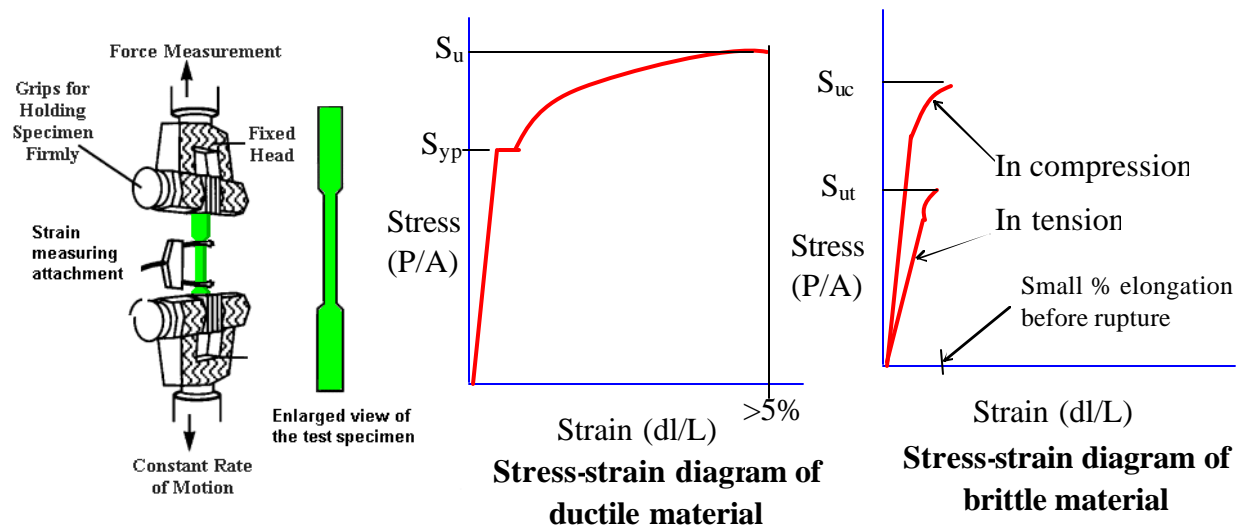
In macro scale, the materials are aggregate of randomly oriented grains. This means that the atomic planes in different grains are randomly oriented (see figure below). **The implication is that, the externally applied force may act as shear force or tensile force to different grains, depending on their orientation of atomic plane.** As a result of this, the microstructure of a material can influence whether there will be slip or a crack from an externally applied force.



Randomly oriented atomic planes may exist in the gross micro-structure

2 Tensile test

A **tensile test**, also known as **uniaxial stress test**, is probably the most fundamental type of mechanical test that can be performed on material to determine its mechanical properties. Tensile tests are simple, relatively inexpensive, and fully standardized. As the material is being pulled apart, its strength, along with how much it will elongate can be easily determined. During this test, instead of plotting the force versus elongation, it is customary to plot the stress versus strain, where stress is force per unit area (P/A), and strain is elongation per unit length (d/L).



Based on the total strain (**or percent elongation**) that a material undergoes before breaking into two pieces during tension test, engineering materials are divided into two distinct groups.

Ductile materials are those that undergo larger strain (more than 5%) before breakage and **brittle materials** are those that break before any appreciable strain (less than 5%).

During tension test, ductile materials remain elastic up to the stress level of **yield strength (S_{yp})**. That is, for the stress level below S_{yp} , the material will go back to its original length after the load is removed. Above S_{yp} stress level, the plastic deformation sets in, that is the material gets permanently deformed, which is unacceptable in mechanical design. During plastic deformation slip occurs between the atomic planes. At a higher stress level, known as **ultimate strength (S_u)** the rupture of the specimen occurs, that is two atomic planes separate out from each other. Obviously, **in the mechanical designing with ductile material, the limiting stress level should be S_{yp}** . Another important characteristic of ductile materials is that the magnitudes of S_{yp} in tension and in compression are very close to each other. That is for ductile materials, **S_{yp} in tension = S_{yp} in compression**. Majority of the commonly used structural metals, such as steel, brass, aluminum and their alloys are ductile material. S_{yp} values for common engineering materials are found in Chapter 14 of the textbook.

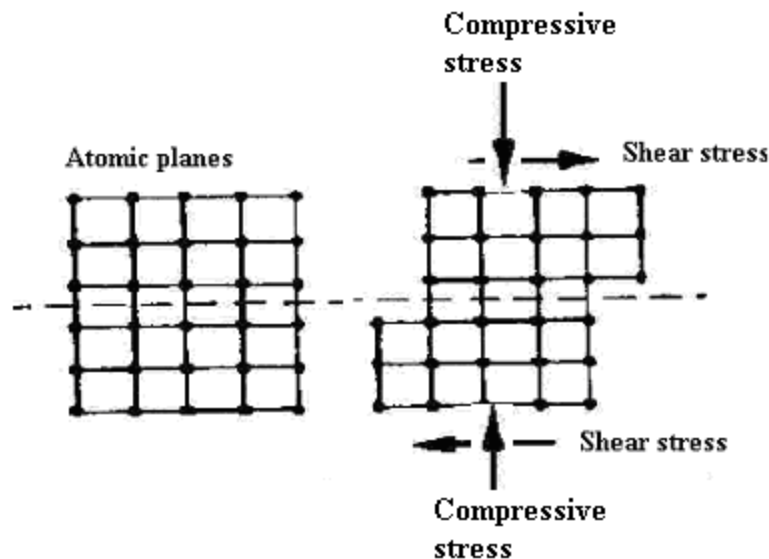
Stress strain characteristics for **brittle materials** are different in two ways. (1) They fail by rupturing (separation of atomic planes) at the ultimate stress (S_u) without any noticeable yielding (slip phenomena) before the rupture. It can be presumed that S_{yp} value of brittle material is greater than S_u (2) Brittle materials are generally stronger in compression than in tension and consequently, **for brittle materials S_u in compression (S_{uc}) is greater than S_u in tension (S_{ut})**. **As a result of this S_{uc} and S_{ut} are the limiting stresses in mechanical design with brittle**

materials. Examples of brittle materials are cast iron, concrete, cast glass etc. S_{uc} and S_{ut} values of different classes of Cast Iron's can be found in Chapter 14 of the textbook.

3. Engineering stress and problem of predicting failure

In a mechanical part, we have seen that due to various types of external loading, a complex three-dimensional state of stress can develop at a point. The most generalized state of stress will have three normal ($\sigma_x, \sigma_y, \sigma_z$) and three shear stresses ($\tau_{xy}, \tau_{yz}, \tau_{zx}$) at a point in three mutually perpendicular directions. The value of each of these six stress components can be negative, zero or positive. Because a specific state of stress can affect the material quite differently than the stress applied in tension test described in section 2 above, the limiting stress levels (S_{yp} or S_u) determined from tension test may not be directly applicable when a complex state of stress exists.

A conceptual model based on the atomic structure can explain this effect for complex stress in influencing the limiting failure stress. For example in if a compressive stress is applied in addition to a shear stress, as shown below, it may take more shear stress to slide an atomic plane over the neighboring atomic plane, as compared to the shear stress in absence of the compressive stress. This is so, because the compressive stress is adding up with the inter-atomic bond forces and thus increases the resistance to slip. Alternatively, if a tensile stress is applied with shear stress, the slip may occur at a lower shear stress.



Thus, when the state of stress is uniaxial tension or compression, the limiting stress values from the tension/compression tests (which are widely published in material handbooks) can be used. But for complex state of stresses, predicting failure is not straightforward.

Several theories of failures have been put forward to predict failure for complex engineering stresses. **Unfortunately, no single failure theory can adequately predict failure in all combination of engineering stresses.**

4. Theories of Failure

We will study four important failure theories, namely (1) maximum shear stress theory, (2) maximum normal stress theory, (3) maximum strain energy theory, and (4) maximum distortion energy theory. Out of these four theories of failure, the **maximum normal stress theory is only applicable for brittle materials, and the remaining three theories are applicable for ductile materials.**

Following are the important common features for all the theories.

- **In predicting failure, the limiting strength (S_{yp} or S_{ut} or S_{uc}) values obtained from the uniaxial testing are used.**
- **The failure theories have been formulated in terms of three principal normal stresses (S_1, S_2, S_3) at a point.** For any given complex state of stress ($\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$), we can always find its **equivalent principal normal stresses (S_1, S_2, S_3).** Thus the failure theories in terms of principal normal stresses can predict the failure due to any given state of stress.
- **The three principal normal stress components $S_1, S_2,$ & S_3 , each which can be comprised of positive (tensile), negative (compressive) or zero value.**
- **When the external loading is uniaxial, that is $S_1 =$ a positive or negative real value, $S_2 = S_3 = 0$, then all failure theories predict the same as that has been determined from regular tension/compression test.**

4.1. Maximum shear stress theory

This theory postulates that failure will occur in a machine part if the magnitude of the maximum shear stress (τ_{max}) in the part exceeds the shear strength (τ_{yp}) of the material determined from uniaxial testing.

If $S_1, S_2,$ & S_3 , are the three principal normal stresses from applied loading, then from Mohr circle, the maximum shear stress in the part is,

$$\tau_{max} = \text{Maximum of the following three quantities } |S_1 - S_2|/2, |S_2 - S_3|/2, \text{ and } |S_3 - S_1|/2$$

In uniaxial testing of the part material, the tensile stress was S_{yp} during yielding. In this case $S_1 = S_{yp}, S_2 = S_3 = 0$. Thus, again from Mohr circle, **shear strength $\tau_{yp} = S_{yp}/2$.**

This theory postulates, that failure will occur when,

$$\tau_{max} = \tau_{yp}$$

$$\text{or } \max \text{ of } [|S_1 - S_2|/2, |S_2 - S_3|/2, \text{ and } |S_3 - S_1|/2] = S_{yp}/2$$

Dividing both side by 2,

$$\max \text{ of } [|S_1 - S_2|, |S_2 - S_3|, \text{ and } |S_3 - S_1|] = S_{yp}$$

Using a design factor of safety N_{fs} , the theory formulates the design equation as,

$$\text{max of } [|S_1 - S_2|, |S_2 - S_3|, \text{ and } |S_3 - S_1|] \text{ should be less than or equal to } S_{yp}/N_{fs}$$

Note: Instead of the above formulation, I find it easier to find the actual τ_{max} from Mohr circle, and then use the following design equation $\tau_{max} = S_{yp}/2N_{fs}$

4.2. Maximum normal stress theory

This theory postulates that failure will occur in a machine part if the maximum normal stress in the part exceeds the normal strength of the material as determined from uniaxial testing.

This theory caters to brittle materials and we have learnt that brittle materials behave differently in compression and tension tests. In compression test it fails when the compressive stress reaches S_{uc} , the ultimate compressive strength of the material and in tensile test it fails when the tensile stress reaches S_{ut} , the ultimate tensile strength of the material. Also generally the magnitude of S_{uc} is larger than S_{ut} for brittle materials.

As the three principal stresses at a point in the part S_1, S_2 or S_3 may be comprised of both tensile and compressive stresses, when this theory is applied, we need to check for failures both from tension and compression. The method of application of this theory is to find the maximum tensile stress, and the maximum compressive stress from the given values of S_1, S_2 and S_3 . The largest positive value among S_1, S_2 & S_3 is the maximum tensile stress and the smallest negative value is the maximum compressive stress. For example if $S_1 = 80$ MPa, $S_2 = -100$ MPa, and $S_3 = -150$ MPa, then the maximum tensile stress = 80 MPa, and the maximum compressive stress = -150 MPa (smallest negative value!).

Thus according to this theory, the safe design condition for brittle material can be given by:

The maximum tensile stress should be less than or equal to S_{ut}/N_{fs} and

The magnitude of the maximum compressive stress should be less than S_{uc}/N_{fs}

4.3. Maximum strain energy theory

This theory postulates that failure will occur when the strain energy per unit volume due to the applied stresses in a part equals the strain energy per unit volume at the yield point in uniaxial testing.

Strain energy is the energy stored in a material due elastic deformation, which is, work done during elastic deformation. Work done per unit volume = strain x average stress.

During tensile test, stress increases from zero to S_{yp} , that is average stress = $S_{yp}/2$.

Elastic strain at yield point = S_{yp}/E , where E is the elastic modulus of elasticity.

Strain energy per unit volume during uniaxial tension = average stress x strain = $S_{yp}^2/2E$

When the applied stress is (S_1, S_2, S_3) then it can be shown (see textbook for derivation) that the strain energy stored in the part = $[S_1^2 + S_2^2 + S_3^2 - 2\mu (S_1S_2 + S_2S_3 + S_3S_1)]/2E$, where μ is Poisson's ratio.

Thus according to this theory, the safe design condition can be given by comparing the two strain energies with a factor of safety:

$$[S_1^2 + S_2^2 + S_3^2 - 2\mu (S_1S_2 + S_2S_3 + S_3S_1)] < (S_{yp}/N_{fs})^2$$

4.4. Maximum distortion energy theory

This theory is also known as shear energy theory or von Mises-Hencky theory.

This theory postulates that failure will occur when the distortion energy per unit volume due to the applied stresses in a part equals the distortion energy per unit volume at the yield point in uniaxial testing.

The total elastic energy due to strain can be divided into two parts. One part causes change in volume, and the other part causes change in shape. Distortion energy is the amount of energy that is needed to change the shape. Derivation of the distortion energy equation can be found in the textbook. Comparing distortion energy for an applied stress (S_1, S_2, S_3) and an applied stress ($S_{yp}, 0, 0$) and using a factor of safety, the following design equation is obtained.

$$S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1 < (S_{yp}/N_{fs})^2$$

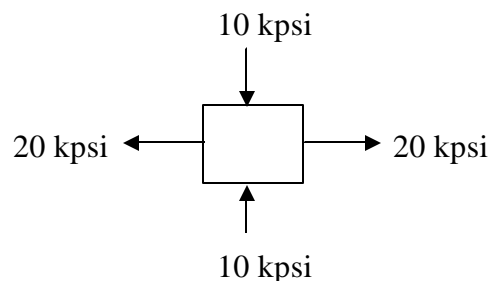
5. Application of Failure theories

- Out of the four theories, only the maximum normal stress theory predicts failure for brittle materials.
- The rest of the three theories are applicable for ductile materials. Out of these three, the distortion energy theory provides most accurate results in majority of the stress conditions. The strain energy theory needs the value of Poisson's ratio of the part material, which is often not readily available. The maximum shear stress theory is conservative.
- For simple unidirectional normal stresses all theories are equivalent, which means all theories will give the same result.

6. Examples

6.1. A part is made of AISI 1020 cold drawn steel. The most critical stress condition at a point in the part is shown.

- What is the factor of safety according to the principal shear stress theory?
- What is the factor of safety according to the von Mises-Hencky theory?



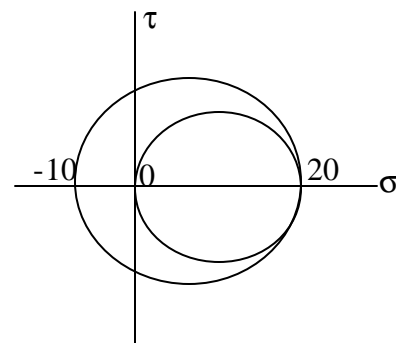
Solution:

From Table 14-4 in the textbook, for cold drawn AISI 1020 steel $S_{yp} = 51$ kpsi

Because there is no applied shear stress, the three principal normal stresses are

$S_1 = 20$ kpsi, $S_2 = -10$ kpsi, and $S_3 = 0$.

- Drawing Mohr circle, Max shear stress developed in the part $\tau_{max} = \text{largest diameter}/2 = 30/2 = 15$ kpsi



Design equation: $\tau_{\max} = S_{yp}/(2N_{fs})$
 or $N_{fs} = S_{yp}/2\tau_{\max} = 51/(2 \cdot 15) = 1.7$

(ii) von Mises-Henkey theory:

$$S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1 < (S_{yp}/N_{fs})^2$$

thus, $N_{fs} = S_{yp}/\text{sqrt of } (S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1)$

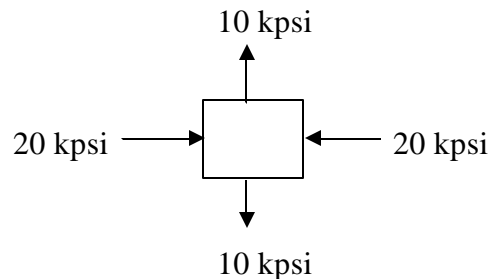
$$= 51/\text{sqrt } [(20^2 + 10^2 + 20 \times 10)]; \text{ note that } -S_1 \cdot S_2 = -20 \times (-10) = 20 \times 10 \text{ \& } S_3 = 0$$

$$= 1.93$$

Comment: two theories for the same stress condition, gave two values of N_{fs} . Max shear stress theory is more conservative and von Mises-Hencky is supposed to be more accurate!

6.2. A part is made of class 25 gray cast iron. The most critical stress condition at a point in the part is shown below. Find the factor of safety.

Solution: The gray cast iron is brittle material.
 Hence we need to apply max normal stress theory.



From table 14-16 in textbook:
 $S_{ut} = 25 \text{ kpsi}$ & $S_{uc} = 100 \text{ kpsi}$

(Comment: no S_{yp} figure is available in the table because it is a brittle material.)

Max compressive stress in the part = 20 kpsi
 So from compressive stress point of view $N_{fsC} = 100/20 = 5$

Max tensile stress in the part = 10 kpsi
 So from tensile stress point of view $N_{fsT} = 25/10 = 2.5$

Thus the factor of safety of the part $N_{fs} = \text{smaller of } (N_{fsC} \text{ \& } N_{fsT}) = 2.5$

7. General problem solving strategy for failure theory

- From external loading, determine the point where the mechanical stress can be most critical. Stress could be combination of axial, bending, transverse shear, or torsional shear stress.
- Determine all normal and shear stresses at that point
- Determine three principal stresses. If there is no shear stress present, then the normal stresses are the principal normal stress. If shear stress is present, you can either use Mohr circle or formula to calculate the principal normal stresses.
- Choose the failure theory applicable.
- Find the S_{yp} or S_{ut} & S_{uc} of the material.
- Apply the chosen theory to determine if factor of safety is adequate.